Rank 1 forms, closed zones and laminae

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Abstract

For a given lattice, we establish an equivalence involving a closed zone of the corresponding Voronoi polytope, a lamina hyperplane of the corresponding Delaunay partition and a rank 1 quadratic form being an extreme ray of the corresponding L-type domain.

An n-dimensional lattice determines two normal partitions of the n space $\mathbf{R}^n$ into polytopes. These are the Voronoi partition and the Delaunay partition. These partitions are dual, i.e. a k-dimensional face of one partition is orthogonal to an $(n-k)$-dimensional face of the other partition. Besides, a vertex of one partition is the center of a polytope of the other partition.

The Voronoi partition consists of Voronoi polytopes with its centers in lattice points. Moreover, any polytope of the Voronoi partition is obtained by a translation of the Voronoi polytope with the center in the origin (=the zero lattice point). Call this polytope *the* Voronoi polytope. It consists of those points of $\mathbf{R}^n$ that are at least as closed to 0 as to any other lattice point.

The Delaunay partition consists of Delaunay polytopes which are, in general, not congruent. The set of all Delaunay polytopes having 0 as a vertex is called the *star* of Delaunay polytopes. Each Delaunay polytope is the convex hull of all lattice points lying on an *empty* sphere. This sphere is called empty, since no lattice point is an interior point of the sphere.

The Voronoi polytope and the Delaunay polytopes of the star are tightly related to minimal vectors of cosets $2L$ in L. A coset Q is called *simple* if it contains, up to sign, only one minimal vector.

For a Delaunay polytope P_D, the lattice vector between any two vertices of P_D is a minimal vector of a coset of $L/2L$. A lattice vector is an edge of a Delaunay polytope of the star (and then, by duality, it defines a facet of the Voronoi polytope) if and only if it is the minimal vector of a simple coset of $L/2L$. All minimal vectors of a non-simple coset are diagonals of a symmetric face of a Delaunay polytope of the star.

The set $\mathcal{P}(P)$ of all faces of all dimensions of a polytope P is partially ordered by inclusion. Call it *face poset of P*. The face poset of the Voronoi polytope P_V of a lattice L determines uniquely the combinatorial structure of P_V and the *L-type* of the Voronoi and Delaunay partitions. The notion of an L-type was introduced by G.Voronoi in [5]. One says that a lattice L *belongs to* or *is of* an L-type if its Voronoi partition has this L-type. In other words, two lattices (and their Voronoi and Delaunay partitions) belong to the same L-type if the corresponding partitions are combinatorially and topologically equivalent, or, equivalently, the face posets of their Voronoi polytopes are isomorphic. If we reverse the order of $\mathcal{P}(P_V)$, we obtain the poset of those faces of Delaunay polytopes of the star that contain the point 0.

There is a small perturbation of a basis of L that does not change the L-type of L. Suppose that a perturbation of the basis changes the L-type. Then the Delaunay partition changes, and there is an empty sphere such that a lattice point either leaves or comes onto the sphere.

There is the following simple but important test of emptiness of a sphere. Let $S \subset \mathbf{R}^n$ be an $(n-1)$-dimensional sphere with the origin point 0 on it. Let $v_1, v_2, ..., v_n$ be n linearly independent lattice vectors with endpoints on S. Let $u \in \mathbf{R}^n$ be an arbitrary vector and $u = \sum_{i=1}^{n} z_i v_i$. (We denote by (p, q) the scalar product of vectors p and q, and set $p^2 = (p, p)$).

Lemma 1 (Proposition 4 of [1]). *The endpoint of u is not an interior point of S if and only if the following inequality holds*

$$u^2 \geq \sum_{i=1}^{n} z_i v_i^2. \tag{1}$$

The endpoint of u lies on S if and only if (1) holds as equality.

Proof. Let c be the center of S. Since the endpoint of v_i lies on S, we have $(v_i - c)^2 = c^2$, $1 \leq i \leq n$, i.e. $v_i^2 = 2(v_i, c)$. Multiplying this equality by z_i,

summing over i and taking in attention that $u = \sum z_i v_i$, we obtain

$$\sum_{i=1}^{n} z_i v_i^2 = 2(u, c).$$

Since the endpoint of u is not an interior point of S, $(u - c)^2 \geq c^2$, i.e. $u^2 \geq 2(u, c)$. Using the above equality, we obtain (1). It is easy to see that (1) holds as equality if and only if the inequality $u^2 \geq 2(u, c)$ holds as equality, i.e. if and only if the endpoint of u lies on S.   □

Any basis $\mathcal{B} = \{b_i, 1 \leq i \leq n\}$ of an n-dimensional lattice L determines uniquely a positive definite quadratic form

$$f(x) = (\sum_{1}^{n} b_i x_i)^2 = \sum_{1 \leq i,j \leq n} a_{ij} x_i x_j.$$

The symmetric matrix a_{ij} of the coefficients of this form is the Gram matrix of the basis $\mathcal{B}$, i.e. $a_{ij} = (b_i, b_j)$. The matrix a_{ij} can be considered as a point of an N-dimensional space, where $N = \binom{n+1}{2}$. In this space, all positive definite forms form an open cone. The closure of this cone is the cone $\mathcal{P}_n$ of all positive semi-definite quadratic forms of order n.

One says that a quadratic form belongs to or is of an L-type if its lattice belongs to this L-type. Hence the cone $\mathcal{P}_n$ is partitioned into L-type domains of forms of the same L-type. Of course, the cone $\mathcal{P}_n$ has many domains of the same L-type corresponding to distinct choices of a basis. Voronoi proved that each L-type domain is an open polyhedral cone of dimension k, $1 \leq k \leq N$. An N-dimensional L-type domain is called *general*. Domains of other dimensions are called *special*. Any face of the closure of a general L-type domain is the closure of a special L-type domain. One-dimensional L-type domains are extreme rays of the closure of a general L-type domain.

Call a form an *edge form* if it belongs to a one-dimensional L-type domain. In [1], an edge form is called *rigid form*, since the only transformation of the corresponding lattice that does not change its L-type is a scaling. A typical edge form is the square of a linear form: $f(x) = (\sum_{i=1}^{n} p_i x_i)^2$, i.e. a rank 1 form. But, for $n \geq 4$, there are edge forms of full rank n.

A polyhedral domain of quadratic forms is called *dicing domain* if all extreme rays of its closure are forms of rank 1. Dicings were defined and studied by Erdahl and Ryshkov [3]. In [3], they give conditions when a dicing domain is an L-type domain and prove the following theorem (Theorem 4.3

of [3]): *An L-type domain is a dicing domain if and only if all the edge forms are rank 1 forms.*

Return to the Voronoi polytope P_V of an n-dimensional lattice L. The Voronoi polytope P_V itself and its facets (i.e. faces of dimension $n - 1$) are centrally symmetric, so as its vertices and edges. But faces of other dimensions are, in general, not centrally symmetric. For example, there are many types of two-dimensional faces: hexagons, and others that are degenerated cases of a hexagon when some of its edges are compressed to a point.

The set of edges of P_V is partitioned into classes of mutually parallel edges. These classes are called *zones*. There are two types of zones: closed and open. A zone is called *closed* if every two-dimensional face contains either two edges of the zone or else none. Otherwise the zone is called *open*. The notions of closed and open zones was introduced by P.Engel in [2].

A closed zone has the following property. Let l be the minimal length of edges of a closed zone Z. Let us shorten all edges of Z onto a value $\varepsilon \leq l$. If $\varepsilon < l$, then Z remains closed, and the new polytope P_V' (with shortened edges) is a Voronoi polytope with the same face poset as P_V. If $\varepsilon = l$, then Z transforms into an open zone, and P_V' has another face poset, since at least one edge vanishes.

Since the Voronoi partition is dual to the Delaunay partition, each edge of a Voronoi polytope is orthogonal to a facet of a Delaunay polytope. A facet F of a Delaunay polytope of a lattice L generates an affine hyperplane H in $\mathbf{R}^n$, namely the hyperplane, where F lies. Obviously F contains n affinely independent lattice points. Hence the intersection $L \cap H$ is an $(n - 1)$-dimensional sub-lattice of L. The Delaunay partition of L generates a partition of the hyperplane H into Delaunay polytopes of the lattice $L \cap H$. It may be that all $(n - 1)$-dimensional Delaunay polytopes of the partition of H are facets of Delaunay polytopes of the original Delaunay partition of L. In this case H is called a *lamina* of the lattice L.

The notion of lamina was introduced and extensively used by Ryshkov and Baranovskii in [4] (see §9.4). If L belongs to a general L-type and if a hyperplane H is not a lamina, then it intersects in an interior point an edge of at least one Delaunay polytope P_D of the star (Lemma 9.3 of [4]). In other words, there are two vertices of P_D that lie in distinct half-spaces determined by H. We reformulate Lemma 9.3 of [4] for a lattice of an arbitrary L-type.

Lemma 2 *A hyperplane is a lamina if and only if it does not intersect any Delaunay polytope of the star in an interior point.*

Obviously, a lamina determines a family of parallel laminae that partitions the lattice L into parallel layers, each of them spanning a lamina. Lemma 2 implies the following corollary.

Corollary *Every Delaunay polytope of a lattice with a lamina lies between two neighboring laminae with vertices on these two laminae.*

The main property of this partition of L into layers, spanning laminae, is that the distances between layers may be changed without changing the L-type of L.

Let H be a hyperplane spanning an $(n-1)$-dimensional sub-lattice of L. Let e be a unit vector orthogonal to H. Then we can define an ϵ-extension along e of the space and of the lattice L as follows. Any vector $v \in \mathbf{R}^n$ is uniquely decomposed as $v = v_e + v_H$, where $v_e = (e, v)e$ and v_H are the projections of v onto e and H, respectively. An ϵ-extension of $\mathbf{R}^n$ along e transforms every vector v into the vector

$$v' = (1 + \epsilon)v_e + v_H = \epsilon(e, v)e + v. \tag{2}$$

Here the ϵ-extension is in fact a contraction if $\epsilon < 0$. In particular, for the norm (=squared length) v'^2 of the extended vector, we obtain the following expression, where we set $\lambda = \epsilon(2 + \epsilon)$:

$$v'^2 = v^2 + \lambda(e, v)^2. \tag{3}$$

Of course, an ϵ-extension of $\mathbf{R}^n$ along e transforms a lattice $L \subset \mathbf{R}^n$ into an *extended lattice* L^ϵ.

There is the following relation between the above introduced notions of a rank 1 form, a closed zone, a lamina and the extended lattice.

Theorem 1 *Let L be an n-dimensional lattice, H be a hyperplane spanning an $(n-1)$-dimensional sub-lattice of L, e be an n-dimensional unit vector orthogonal to H. Let $f(x)$ be the quadratic form corresponding to a basis $\{b_i : 1 \leq i \leq n\}$ of L and $D(f)$ be the L-type domain of f. The following assertions are equivalent:*

(i) H is a lamina of the Delaunay partition of L;

(ii) the Voronoi polytope P_V of L has a closed zone Z_e of edges parallel to the vector e;

(iii) the ϵ-extended along e lattice L^ϵ has the same L-type as L for all $\epsilon > 0$;

(iv) the rank 1 form $f_e(x) = (e, \sum_1^n b_i x_i)^2$ lies on an extreme ray of the closure of $D(f)$, i.e. $f + \lambda f_e \in D(f)$ for all nonnegative λ.

Proof. (i)$\Rightarrow$(ii). Let H be a lamina partitioned into facets of Delaunay polytopes of L. We can suppose that H contains the origin 0. Consider the edges of the Voronoi polytope P_V that are orthogonal to the lying in H facets of the star. Obviously these edges are parallel to e and form the zone Z_e. We show that Z_e is closed. If not, there is a 2-face T of P_V containing exactly one edge $u_1 \in Z_e$. The edges of T form a polygon. Let $u_1, u_2, ..., u_k$ be consecutive edges of this polygon. Let F_i be the facet of the star that is orthogonal to the edge u_i, $1 \leq i \leq k$. The set of facets $\{F_i : 1 \leq i \leq k\}$ has a common $(n-2)$-dimensional face of the star that is orthogonal to T and lies in the lamina H. Since, for $2 \leq i \leq k$, the edge u_i is not parallel to u_1, the facet F_i does not lie in the lamina H. Hence there is an index j such that the facets F_j and F_{j+1} lie in distinct halfspaces separated by H. Obviously, F_j and F_{j+1} are facets of a same Delaunay polytope P_j of the star, and H intersects P_j. This contradicts to definition of a lamina. Hence Z_e cannot be open.

(ii)$\Rightarrow$(i). Let $u_1 \in Z_e$. Consider the facet F_1 of the star that is orthogonal to u_1 and contains $0 \in L$. F_1 spans a hyperplane H that is orthogonal to e and contains $0 \in L$. Let T_1 be a 2-face of P_V containing u_1, and u_2 be the second edge from Z_e contained in T_1. Let F_2 be the facet of the star that is dual to u_2. F_2 intersects F_1 by an $(n-2)$-face that is dual to T_1 and contains 0. Hence F_2 contains 0 and is orthogonal to e. This implies that F_2 lies in H. Similarly, for $i = 3, 4, ...$, we consider the 2-face T_i containing $u_i, u_{i+1} \in Z_p$, and prove that the facet F_i of the star dual to u_i lies in H. Since P_V is a polytope, it has a finite number of edges. Hence there is i_0 such that $u_{i_0} = u_1$. We obtain a set of facets of the star lying in H such that the facet F_i intersects F_{i-1} and F_{i+1}. Therefore the intersection of the star with H consists of facets of the star. Since this is true for all Voronoi polytopes having centers in H, the hyperplane H is partitioned into facets of Delaunay polytopes. This means that H is lamina.

(i)$\Rightarrow$(iii). It is sufficient to prove that no lattice point comes onto or leaves the empty sphere S of a Delaunay polytope P_D. Without loss of generality, we can suppose that P_D belongs to the star. Let $v_1, v_2, ..., v_n$ be n linearly

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independent lattice vectors with endpoint in vertices of P_D, i.e. they lie on the sphere S. Let u be any lattice vector and let $u = \sum_{i=1}^{n} z_i v_i$ be its decomposition by v_i, $1 \leq i \leq n$. According to (2), after an ϵ-extension of the space along e, u and v_i are transformed into the vectors

$$u' = u + \epsilon(u, e)e, \ v'_i = v_i + \epsilon(v_i, e)e.$$

By Lemma 1, it is sufficient to prove that the inequality $u'^2 \geq \sum_{i=1}^{n} z_i v'^2_i$ is strict or is an equality according to the inequality $u^2 \geq \sum_{i=1}^{n} z_i v_i^2$ is strict or is an equality.

Recall that H contains an $(n-1)$-dimensional sub-lattice. Hence $(v, e) = k(v)\alpha$ for any lattice vector v, where $k(v)$ is an integer and α does not depend on v. Let $k = k(u)$ and $k_i = k(v_i)$. Multiplying the equality $u = \sum_{i=1}^{n} z_i v_i$ by e, we obtain the equality $k = \sum_{i=1}^{n} z_i k_i$. By Corollary of Lemma 2, the vertices of P_D lie on two neighboring laminae. Hence $|k_i| = 0, 1$. Without loss of generality we can suppose that $k_i \geq 0$, i.e. $k_i = 0, 1$ and therefore $(v_i, e)^2 = k_i^2 = k_i$. Hence, using (3), we have

$$v'^2_i = v_i^2 + \lambda(v_i, e)^2 = v_i^2 + \lambda k_i \alpha^2.$$

Since $\sum_{i=1}^{n} z_i k_i = k$, we obtain the equality

$$\sum_{i=1}^{n} z_i v'^2_i = \sum_{i=1}^{n} z_i v_i^2 + \lambda \alpha^2 \sum_{i=1}^{n} z_i k_i = \sum_{i=1}^{n} z_i v_i^2 + \lambda \alpha^2 k. \tag{4}$$

Suppose that $u^2 > \sum_{i=1}^{n} z_i v_i^2$. We show that then $\sum_{1}^{n} z_i v'^2_i < u'^2 = u^2 + \lambda(u, e)^2 = u^2 + \lambda \alpha^2 k^2$. For an integer k, we have $k \leq k^2$ with equality if and only if $k = 0, 1$. The above inequalities and the equality (4) imply

$$\sum_{1}^{n} z_i v'^2_i < u^2 + \lambda \alpha^2 k^2 = u'^2.$$

Now let $u^2 = \sum_{1}^{n} z_i v_i^2$. This means that u has the endpoint in a vertex of P_D. Hence $k = 0, 1$, i.e. $k^2 = k$. Hence the equality (4) implies the equality

$$\sum_{1}^{n} z_i v'^2_i = u^2 + \lambda \alpha^2 k^2 = u'^2.$$

(iii)$\Rightarrow$(iv). We prove in the implication (i)$\Rightarrow$(iii) that $f(L^\epsilon) \in D(f)$ for every $\epsilon \geq 0$. Now we show that

$$f(L^\epsilon) = f(L) + \lambda f_e,$$

where $\lambda = \epsilon(2 + \epsilon)$. The basic vectors of the extended lattice L^ϵ have the form

$$b'_i = b_i + \epsilon(e, b_i)e, \quad 1 \leq i \leq n.$$

Hence the coefficients a'_{ij} of the quadratic form $f^\epsilon = f(L^\epsilon)$ are as follows

$$a'_{ij} = (b'_i, b'_j) = (b_i, b_j) + \lambda(e, b_i)(e, b_j).$$

Hence we obtain

$$f^\epsilon(x) = f(x) + \lambda(\sum_{i=1}^n (e, b_i)x_i)^2 = f(x) + \lambda f_e(x).$$

This means that the ray $\{\lambda f_e : \lambda \geq 0\}$ belongs to the closure of $D(f)$. Since this ray is a one-dimensional L-type domain, it is an extreme ray of cl$D(f)$.

(iv)$\Rightarrow$(iii). If f_e lies on an extreme ray of cl$D(f)$, then the quadratic function $f^\epsilon = f + \epsilon(2 + \epsilon)f_e$ belongs to $D(f)$ for all $\epsilon \geq 0$. The matrix a'_{ij} of f^ϵ is

$$a_{ij} + \epsilon(2 + \epsilon)(e, b_i)(e, b_j) = (b'_i, b'_j).$$

Hence f^ϵ is a quadratic form of the ϵ-extended lattice L^ϵ. So, L^ϵ has the same L-type as L for all $\epsilon \geq 0$.

(iii)$\Rightarrow$(i). We show that if the extended along e lattice L^ϵ has the same L-type as L for all $\epsilon > 0$, then the hyperplane H which is orthogonal to e is a lamina. Suppose H is not a lamina. Then H intersects a Delaunay polytope P_D of the star in an interior point. Hence there are two vectors v_1 and v_2 with endpoints in vertices of P_D such that $k_1 = k(v_1) \geq 1$ and $k_2 = k(v_2) \leq -1$. Consider the lattice vector

$$u = q(k_1 v_2 - k_2 v_1),$$

where the integer q is chosen such that the endpoint of u does not lie on the empty sphere S circumscribing P_D. Expand the pair of vectors v_1, v_2 up to a set of n independent vectors with endpoints in vertices of P_D. Then the above expression for u is the representation of u as a linear combination of

these n independent vectors. Since the endpoint of u does not lie on S, by Lemma 1, $\Delta \equiv u^2 - q(k_1 v_2^2 - k_2 v_1^2) > 0$.

Consider an ϵ-extension of the space along e. Let u', v_1', v_2' be the ϵ-extended vectors. Consider the difference $\Delta' = u'^2 - q(k_1 v_2'^2 - k_2 v_1'^2)$. Using (3), we obtain

$$\Delta' = \Delta + \lambda(e, u)^2 - q(k_1 \lambda(e, v_2)^2 - k_2 \lambda(e, v_1)^2).$$

Since $(e, u) = \alpha k(u) = 0$, $(e, v_1) = \alpha k_1$, $(e, v_2) = \alpha k_2$ and $k_2 < 0$, we have

$$\Delta' = \Delta - \lambda q k_1 |k_2|(|k_2| + k_1).$$

Let $p = q k_1 |k_2|(k_1 + |k_2|) > 0$. Then for $\lambda = \frac{\Delta}{p}$, we obtain $\Delta' = 0$. Lemma 1 implies that, for this λ, the endpoint of u lies on S. This means that there is $\epsilon > 0$ such that the ϵ-extended lattice L^ϵ has the L-type distinct from the L-type of L. We obtain a contradiction. $\qquad\square$

Remark 1. The rank 1 function $f_e(x) = (e, \sum_1^n b_i x_i)^2$ depends on the basis corresponding to the function $f(x)$. But this dependence is not essential in the following sense. Recall that $(e, b_i) = \alpha k(b_i)$, where $k_i = k(b_i)$ is an integer. If we slightly move $f(x)$ in the domain $D(f)$, then we slightly change the basis $\mathcal{B} = \{b_i : 1 \le i \le n\}$ and the scalar products $(e, b_i) = \alpha k_i$. But, since k_i is an integer, it cannot change slightly. Hence only α slightly changes. This implies that this movement of $f(x)$ inside of the L-type domain $D(f)$ causes a movement of $f_e(x) = \alpha^2(\sum_1^n k_i x_i)^2$ along the ray $\{\lambda(\sum_1^n k_i z_i)^2 : \lambda \ge 0\}$. The collections of the integers $\{k_i : 1 \le i \le n\}$ is an invariant of the L-type domain $D(f)$.

Remark 2. The equivalence (i)$\Leftrightarrow$(iv) of Theorem 1 is mentioned in the paper [3]. After the proof of Theorem 4.3, the authors of [3] write:

> The ideas used in this proof can be extended to cover the case in which only a portion of the edge forms have rank 1. For such an L-type domain each rank 1 edge form can be associated with a D-family of parallel hyperplanes G, and the L-partitions $\mathcal{S}$ of lattices on this domain are refinement of the partition determined by G. In the other direction, any hyperplane which does not intersect the interior of any L-polytope of an L-partition can be associated with a rank 1 edge form of the corresponding L-type

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domain. Such hyperplanes are members of a D-family of parallel hyperplanes.

Note that here an L-partition and an L-polytope mean a Delaunay partition and a Delaunay polytope, and a D-family is a family of parallel laminae.
In fact, the authors asserts that the ideas used in the proof of Theorem 4.3 of [3] can be extended to the proof of the equivalence (i)⇔(iv). But it seems to us that the proof given above is not complete.

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